Math 564: Advance Analysis 1
Lecture 11

Borel Isomorphism Theorem. Any two unctbl Polish spaces are Borel isomorphic. In particular, they are all of cardinality continuum and Borel isomorphic to the Cantor space $2^{\mathbb{N}}$.

Def. A *standard Borel space* is a measurable space $(X, \mathcal{B})$, where $\mathcal{B}$ is the σ-alg of Borel sets for some Polish top. on X. In other words, a standard Borel space is a Polish space where we forgot the topology but kept the Borel sets.

The Borel Isom. theorem says that there is only one, up to isomorphism, unctbl standard Borel space.

Def. A *standard probability space* is a prob. space $(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B})$ is standard Borel (i.e. a Polish space with a Borel prob. measure on it).

Prop. For a 2nd ctbl Hausdorff top space X (e.g. a separable metric space), for any Borel probability measure μ on X, the atoms of μ are singletons.

Proof. Just König's lemma + pigeons. HW.

Cor. Let (X, μ) be as in the above proposition, then $\mu = \mu' + \sum_{n \in \mathbb{N}} a_n \delta_{x_n}$, where $\{x_n : n \in \mathbb{N}\} \subseteq X$, $a_n \geq 0$, and μ' is atomeless.

Proof. Since atoms are disjoint singletons, there are only ctbly many of them (by the pre-measure exhaustion lemma), so we can remove them from μ to obtain μ'. $\square$

Measure isomorphism theorem. Any two atomless standard prob. spaces are measure-isomorphic. In fact, there is a (genuine) Borel isomorphism between them that is a measure-isomorphism. In particular, all such spaces are isomorphic to $([0,1], \lambda)$.

We now give ideas of proofs of each of these theorems.

Borel isom. proof-sketch. We fix an unctbl Polish space X and show that it is Borel isom. to $2^{\mathbb{N}}$. We do so by proving

(a) $X \overset{\text{Borel}}{\hookrightarrow} 2^{\mathbb{N}}$.

(b) $2^{\mathbb{N}} \overset{\text{Borel}}{\hookrightarrow} X$.

$\overset{\text{Borel}}{\hookrightarrow}$ means a Borel embedding.

This is enough by the proof of Schröder–Bernstein theorem because the latter only uses images and preimages of Borel sets under the given Borel embeddings.

Proof of (a). We build a "binary representation" map for X as follows. Fix a ctbl open basis $(U_n)_{n\in\mathbb{N}}$ for the topology. Define $b : X \to 2^{\mathbb{N}}$ by $x \mapsto (\mathbb{1}_{U_n}(x))_{n\in\mathbb{N}}$. Since X is Hausdorff, b is injective. Preimage of a cylinder is a fin. intersection of sets of the form U_n or U_n^c, so b is Borel. By the Luzin–Souslin thm from Descriptive Set Theory, b is a Borel embedding.

Proof of (b). By the Cantor–Bendixson theorem from DST, $X = P \sqcup U$, where P is a closed subset of X that is perfect (no isolated points) and U is ctbl open set. Cantor's Perfect Set Theorem says that $2^{\mathbb{N}}$ homeomorphically embeds into every nonempty perfect Polish space, so if X is unctbl, P is nonempty perfect Polish, so $2^{\mathbb{N}} \hookrightarrow X$, i.e. topologically embeds into X. $\square$

Measure isom. proof-sketch. We prove that for every atomless standard prob space (X, μ), there is a Borel isom with $([0,1], \lambda)$ that is also a measure isomorphism. Because (X,μ) has no atoms (which are singletons), X has to be unctbl. There is a Borel isom. $f: X \to [0,1]$. Then $\mu' := f_* \mu$ is an atomless Borel prob. measure on $[0,1]$. In other words, we have assumed from the beginning that $X = [0,1]$ and μ is a Borel measure on $[0,1]$.

By a HW question, μ and λ on $[0,1]$ are measure-isomorphic, and building a Borel isomorphism witnessing this is done via the Borel isomorphism theorem, which helps sweeping a null sets under the rug. $\square$

Integration. Let $(X, \mathcal{S})$ be a measurable space. Denote by $L(X,\mathcal{S})$ and $L^+(X,\mathcal{S})$ the sets of $\mathcal{S}$-measurable functions to $[-\infty, \infty]$ and to $[0, \infty]$, respectively.

The $0 \cdot \infty$ convention. In the measure-theoretic context, $0 \cdot \infty$ is 0.

Note that $L^+ := L^+(X, \mathcal{S})$ is closed under non-negative linear combinations and multiplication. An integral $\int$ on L^+ is a nonnegative linear functional on L^+ that is ctbl-additive, i.e.

(i) $\int (a \cdot f + b \cdot g) = a \cdot \int f + b \cdot \int g$ for all $a, b \geq 0$ and $f, g \in L^+$.

(ii) $\int f \geq 0$, in particular, if $f \leq g$ then $\int f \leq \int g$, for all $f, g \in L^+$.

(iii) $\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$ for all $f_n \in L^+$.

Obs. Each integral $\int$ on L^+ defines a measure $\mu_\int$ on $(X, \mathcal{S})$ by
$$A \mapsto \int \mathbb{1}_A,$$
for each $A \in \mathcal{S}$, where $\mathbb{1}_A$ is the indicator function of A.

Our goal is to reverse this: given a measure μ on X, we build an integral $\int_\mu$ s.t. $\mu_{\int_\mu} = \mu$.

The general real-valued function will be handled afterwards by the realization that any such function $f: X \to \mathbb{R}$ splits into a difference $f_+ - f_-$ of nonnegative functions, called its positive and negative parts, defined by $f_+ := f|_{f^{-1}([0,\infty))}$ and $f_- := -f|_{f^{-1}((-\infty,0])}$. Note that f_+ & f_- have disjoint supports (i.e. they are nonzero on disjoint sets).

Given μ, we first define the integral $\int$ on the so-called simple functions, and then we show that every func. in L^+ is approximated from below by simple functions, so we extend $\int$ to L^+.

Simple functions. A simple function $f: X \to \mathbb{R}$ is just a linear combination of indicator functions of μ-measurable sets (hence they are μ-measurable).

Obs. A function is simple $\Leftrightarrow$ it is measurable and has finite image.
Proof. $\Rightarrow$. Let $f = \sum_{i<n} a_i \cdot \mathbb{1}_{A_i}$, then $f(X) \subseteq \{\sum_{i<n} b_i a_i : \vec{b} \in 2^n\}$.

$\Leftarrow$. Suppose $f(X) = \{a_0, \dots, a_{n-1}\}$ then $f = \sum_{i<n} a_i \cdot \mathbb{1}_{A_i}$, where $A_i := f^{-1}(a_i)$. This is called the standard representation of f. $\square$

Note that simple functions form an $\mathbb{R}$-algebra, i.e. it is an $\mathbb{R}$-vector space closed under multiplication (indeed, $f \cdot g$ is simple if f, g are because $\mathbb{1}_A \cdot \mathbb{1}_B = \mathbb{1}_{A \cap B}$).